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Virasoro operators in the continuous basis of string field theory

Ehud Fuchs, Michael Kroyter and Alon Marcus

School of Physics and Astronomy
The Raymond and Beverly Sackler Faculty of Exact Sciences
Tel Aviv University, Ramat Aviv, 69978, Israel
E-mail: udif@tau.ac.il, mikroyt@tau.ac.il, alon@tau.ac.il

Abstract: In this work we derive two important tools for working in the κ basis of string field theory. First we give an analytical expression for the finite part of the spectral density ρ_{fin}. This expression is relevant when both matter and ghost sectors are considered. Then we calculate the form of the matter part of the Virasoro generators L_n in the κ basis, which construct string field theory's derivation Q_{BRST}. We find that the Virasoro generators are given by one dimensional delta functions with complex arguments.

Keywords: String Field Theory, Conformal and W Symmetry.

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1. Introduction and summary

String field theory [1] is a non-perturbative, off-shell formulation of string theory. The basic entities of this theory are fields over the space of string configurations. These fields are multiplied using a non-commutative star product, defined by the gluing of the right half of one string to the left half of the other. The derivation operator on these fields, that appears in the kinetic term, is Q_{BRST}.

Gross and Jevicki [2] gave an explicit formulation of the star product in the (infinite dimensional) oscillator basis. They defined the star product by an interaction vertex, which is a squeezed state over a direct product of three Fock spaces. This form of the vertex implies that squeezed states form a subalgebra of the star product.

In [3] Rastelli Sen and Zwiebach diagonalized these matrices and found their spectrum. The eigenvectors of these matrices form a continuous basis whose eigenvalues are in the range $-\infty < \kappa < \infty$. Working in this basis simplifies calculations involving the star product.

Calculating normalization of squeezed states in the continuous basis involves determinants of continuous matrices. If these matrices are diagonal $X = X_\kappa \delta(\kappa - \kappa')$, then the determinant has the form

$$\det X = \exp\left(\operatorname{tr}\log X\right) = \exp\left(\int d\kappa \delta(\kappa - \kappa) \log X_\kappa\right). \tag{1.1}$$

A similar expression holds for matrices in the $\mathcal{H}_{\kappa^2}$ subalgebra [4]. The delta function is the spectral density $\rho(\kappa)$ of the continuous basis. This spectral density diverges, and in the level-truncation regularization its divergence is κ independent and behaves as $\rho^L(\kappa) = \frac{\log L}{2\pi}$. Because of this divergence, it seems that the determinant can only get the values $0, 1, \infty$ depending on whether the integral $\int d\kappa \log X$ is negative, zero or positive. But $\rho(\kappa)$ has a finite contribution $\rho_{\text{fin}}(\kappa)$ which is κ dependent. When the ghost sector contributions cancel these infinities, $\rho_{\text{fin}}(\kappa)$ cannot be ignored, and the expression

$$\exp\left(\int d\kappa \rho_{\text{fin}}(\kappa) \log X\right),$$

should be taken into account. Belov and Konechny calculated $\rho_{\text{fin}}(\kappa)$ numerically in [5]. We find the analytic expression

$$\rho_{\text{fin}}(\kappa) = \frac{4\log(2) - 2\gamma - \Psi\left(\frac{i\kappa}{2}\right) - \Psi\left(-\frac{i\kappa}{2}\right)}{4\pi}, \tag{1.2}$$

where γ is Euler's constant, and Ψ is the digamma (polygamma) function.

Another missing ingredient of the continuous basis is the form of the Virasoro generators. The Virasoro generators are used in string field theory to construct Q_{BRST}, from which the kinetic term around the perturbative vacuum is built. Other derivations built from the Virasoro generators can serve as kinetic terms as well [6]. The Virasoro generators are also useful in the construction of surface states [7], and in particular of surface state projectors [8]. In [9] it was noticed that the expression for L_0 in the continuous basis diverges. This is true for all the Virasoro generators. Nonetheless, we manage to find an analytic expression for them. For a single scalar field, L_0 is given by

$$L_0 = \alpha' p_0^2 + \sum_{n=1}^{\infty} n a_n^\dagger a_n = \alpha' p_0^2 + \int_{-\infty}^{\infty} \frac{d\kappa\, d\kappa'\, a_{\kappa'}^\dagger a_\kappa}{\sqrt{\mathcal{N}(\kappa)\mathcal{N}(\kappa')}} g_0^{\kappa,\kappa'}, \tag{1.3}$$

$$g_0^{\kappa,\kappa'} \equiv \sum_{n=1}^{\infty} n v_n^\kappa v_n^{\kappa'} = \frac{\cosh\left(\frac{(\kappa+\kappa')\pi}{4}\right)}{2} \big(\delta(\kappa-\kappa'+2i) + \delta(\kappa-\kappa'-2i)\big). \tag{1.4}$$

The use of complex arguments in the one dimensional delta function is somewhat unorthodox, but the definition of the delta function for complex arguments is essentially the same as for real arguments. We elaborate on the definition of the delta function and demonstrate its use in the body of the paper.

The complex delta functions hide the high divergence of L_0 in the continuous basis. They appear in all the Virasoro generators. The midpoint preserving reparameterization generators $K_n = L_n - (-1)^n L_{-n}$ should have milder divergences [10], and indeed they do not contain complex delta functions.

This paper is organized as follows. In section 2 we fix our notations for the continuous basis and give a useful integral expression for the eigenvectors v_n^κ. In section 3 we calculate the finite part of the spectral density. In section 4 we introduce a useful operator, which we denote $\mathcal{L}_\kappa$, and elaborate on the definition and use of the one dimensional delta function with a complex argument. We use these tools in section 5 to calculate the Virasoro generators. Sections 4 and 5 are independent of section 3.

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2. The continuous basis

In this section we summarize the properties of the continuous basis, and set our notations. In [3] the star algebra derivation $K_1 = L_1 + L_{-1}$ was represented by a matrix over the oscillators basis $[K_1, v \cdot a] = (\mathrm{K}_1 v) \cdot a$ and its spectrum was found to be

$$\mathrm{K}_1 v^\kappa = \kappa v^\kappa \,, \tag{2.1}$$

where $-\infty < \kappa < \infty$ and v^κ are the eigenvectors of the matrix K_1. The generating function of these eigenvectors is

$$f_\kappa(z) \equiv \sum_{n=1}^{\infty} \frac{v_n^\kappa}{\sqrt{n}} z^n = \frac{1}{\kappa}\left(1 - e^{-\kappa \tan^{-1} z}\right). \tag{2.2}$$

This relation can be inverted to give

$$v_n^\kappa = \frac{\sqrt{n}}{2\pi i} \oint \frac{f_\kappa(z)}{z^{n+1}} dz \,. \tag{2.3}$$

Okuyama [11] found the orthogonality and completeness relations of the eigenvectors

$$\sum_{n=1}^{\infty} v_n^\kappa v_n^{\kappa'} = \mathcal{N}(\kappa)\delta(\kappa - \kappa') \,, \qquad \int_{-\infty}^{\infty} \frac{d\kappa}{\mathcal{N}(\kappa)} v_n^\kappa v_m^\kappa = \delta_{n,m} \,, \tag{2.4}$$

where the normalization is given by

$$\mathcal{N}(\kappa) = \frac{2}{\kappa} \sinh\left(\frac{\kappa\pi}{2}\right). \tag{2.5}$$

The continuous basis creation and annihilation operators are given by

$$a_\kappa^\dagger = \sum_{n=1}^{\infty} \frac{v_n^\kappa}{\sqrt{\mathcal{N}(\kappa)}} a_n^\dagger \,, \qquad a_\kappa = \sum_{n=1}^{\infty} \frac{v_n^\kappa}{\sqrt{\mathcal{N}(\kappa)}} a_n \,, \tag{2.6}$$

where a_n are the usual modes of the string. The inverse relations are

$$a_n^\dagger = \int_{-\infty}^{\infty} d\kappa \frac{v_n^\kappa}{\sqrt{\mathcal{N}(\kappa)}} a_\kappa^\dagger \,, \qquad a_n = \int_{-\infty}^{\infty} d\kappa \frac{v_n^\kappa}{\sqrt{\mathcal{N}(\kappa)}} a_\kappa \,. \tag{2.7}$$

With these relations the κ oscillators satisfy the canonical commutation relations

$$[a_\kappa, a_{\kappa'}^\dagger] = \delta(\kappa - \kappa') \,. \tag{2.8}$$

We end this section by giving an integral representation for v_n^κ [4] that is more useful then (2.3) in some calculations.

$$\begin{aligned} v_{2n-1}^\kappa &= \frac{(-1)^{n+1}\sqrt{2n-1}}{\pi} \mathcal{N}(\kappa) \int_1^\infty \frac{\cos(\kappa \coth^{-1}(x))}{x^{2n}} dx \\ &= \frac{(-1)^{n+1}\sqrt{2n-1}}{\pi} \mathcal{N}(\kappa) \int_0^\infty \frac{\cos(\kappa u)\tanh^{2n-2}(u)}{\cosh^2(u)} du \,, \end{aligned} \tag{2.9}$$

where we substituted $x = \coth(u)$ in the last step. The analogous expression for v_{2n} reads

$$\begin{aligned} v^{\kappa}_{2n} &= \frac{(-1)^n\sqrt{2n}}{\pi}\mathcal{N}(\kappa)\int_1^\infty \frac{\sin\left(\kappa\coth^{-1}(x)\right)}{x^{2n+1}}dx \\ &= \frac{(-1)^n\sqrt{2n}}{\pi}\mathcal{N}(\kappa)\int_0^\infty \frac{\sin(\kappa u)\tanh^{2n-1}(u)}{\cosh^2(u)}du\,. \end{aligned} \tag{2.10}$$

Collectively they can be written as

$$v^{\kappa}_n = \frac{i^{n-1}\sqrt{n}}{2\pi}\mathcal{N}(\kappa)\int_{-\infty}^\infty \frac{e^{i\kappa u}\tanh^{n-1}(u)}{\cosh^2(u)}du\,. \tag{2.11}$$

These expressions will allow us to change the order of summation and integration in the calculations of the spectral density and the Virasoro generators.

3. The spectral density

The spectral density $\rho(\kappa)$ is needed for calculating traces and determinants of operators that are diagonal in the continuous basis. Ignoring for a moment the fact that the trace is base invariant, we can calculate

$$\mathrm{tr}G = \langle n|\,G\,|n\rangle = \langle n|\,\kappa\rangle\,\langle\kappa|\,G\,|\kappa'\rangle\,\langle\kappa'|\,n\rangle\,. \tag{3.1}$$

Now, using the fact that G is diagonal in the κ basis

$$\langle\kappa|\,G\,|\kappa'\rangle = G_\kappa\delta(\kappa-\kappa')\,, \tag{3.2}$$

we get

$$\mathrm{tr}G = \langle n|\,\kappa\rangle\,\langle\kappa|\,n\rangle\,G_\kappa = \rho(\kappa)G_\kappa\,, \tag{3.3}$$

where we define the spectral density

$$\rho(\kappa) \equiv \langle\kappa|\,n\rangle\,\langle n|\,\kappa\rangle = \frac{1}{\mathcal{N}(\kappa)}\sum_{n=1}^\infty v^\kappa_n v^\kappa_n\,. \tag{3.4}$$

From the invariance of the trace $\mathrm{tr}G = \langle\kappa|\,G\,|\kappa\rangle$, it is obvious that $\rho(\kappa)$ diverges like $\delta(0)$. In [3] it was shown that the leading term in level truncation regularization is $\rho^L(\kappa) = \frac{\log L}{2\pi}$, where L denotes the level. This term is κ independent. In [5] the finite, κ dependent, contribution to the spectral density was defined as

$$\rho^{2L}_{\mathrm{fin}}(\kappa) = \frac{1}{\mathcal{N}(\kappa)}\sum_{n=1}^{2L} v^\kappa_n v^\kappa_n - \frac{1}{2\pi}\sum_{n=1}^{L}\frac{1}{n}\,, \tag{3.5}$$

and evaluated numerically. In this section we obtain an analytical expression for this term

$$\rho_{\mathrm{fin}}(\kappa) = \lim_{L\to\infty}\rho^{2L}_{\mathrm{fin}}(\kappa)\,. \tag{3.6}$$

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We begin by regularizing the two diverging sums on the r.h.s of (3.5) with powers of the variable z to get

$$\rho_{\rm fin}(z,\kappa) = \frac{1}{\mathcal{N}(\kappa)}\sum_{n=1}^{\infty}\left(v_{2n-1}^{\kappa}v_{2n-1}^{\kappa}z^n + v_{2n}^{\kappa}v_{2n}^{\kappa}z^n\right) - \frac{1}{2\pi}\sum_{n=1}^{\infty}\frac{z^n}{n}\,, \tag{3.7}$$

where $|z|<1$. The last sum gives

$$-\frac{1}{2\pi}\sum_{n=1}^{\infty}\frac{z^n}{n} = \frac{1}{2\pi}\log(1-z)\,. \tag{3.8}$$

We evaluate the first sum by using eq. (2.9)

$$\begin{aligned}
\frac{1}{\mathcal{N}(\kappa)}\sum_{n=1}^{\infty}v_{2n-1}^{\kappa}v_{2n-1}^{\kappa}z^n &= \frac{2z^{\frac{3}{2}}}{\mathcal{N}(\kappa)}\partial_z\sum_{n=1}^{\infty}\frac{v_{2n-1}^{\kappa}v_{2n-1}^{\kappa}}{2n-1}z^{n-\frac{1}{2}} \\
&= \frac{2\mathcal{N}(\kappa)z^{\frac{3}{2}}}{\pi^2}\partial_z\int_0^{\infty}du\,dv\frac{\sqrt{z}\cos(\kappa u)\cos(\kappa v)}{\cosh^2(u)\cosh^2(v)}\times \\
&\qquad\qquad\times\sum_{n=1}^{\infty}\left(z\tanh^2(u)\tanh^2(v)\right)^{n-1} \\
&= \frac{2\mathcal{N}(\kappa)z^{\frac{3}{2}}}{\pi^2}\partial_z\int_0^{\infty}\frac{\sqrt{z}\cos(\kappa u)\cos(\kappa v)}{\cosh^2(u)\cosh^2(v)-z\sinh^2(u)\sinh^2(v)}du\,dv \\
&= \frac{z}{\pi}\int_0^{\infty}\frac{\cos(\kappa v)\cos(\kappa\tanh^{-1}(\sqrt{z}\tanh(v)))}{\cosh^2(v)-z\sinh^2(v)}dv\,.
\end{aligned} \tag{3.9}$$

In the last step we replaced $\int_0^\infty du\cos(\kappa u)\cdots$ with $\frac{1}{2}\int_{-\infty}^{\infty}du\exp(i\kappa u)\cdots$ and evaluated the u integral by closing the contour in the upper half plane for $\kappa>0$ (with analytic continuation for $\kappa\le 0$) picking up the residues at $u=\frac{(2n-1)\pi i}{2}\pm\tanh^{-1}(\sqrt{z}\tanh(v))$.

In a similar way we evaluate the second term on the r.h.s of (3.7)

$$\begin{aligned}
\frac{1}{\mathcal{N}(\kappa)}\sum_{n=1}^{\infty}v_{2n}^{\kappa}v_{2n}^{\kappa}z^n &= \frac{2z}{\mathcal{N}(\kappa)}\partial_z\sum_{n=1}^{\infty}\frac{v_{2n}^{\kappa}v_{2n}^{\kappa}}{2n}z^n \\
&= \frac{2}{\pi^2}\mathcal{N}(\kappa)z\partial_z\int_0^{\infty}\frac{z\sin(\kappa u)\sin(\kappa v)\tanh(u)\tanh(v)}{\cosh^2(u)\cosh^2(v)-z\sinh^2(u)\sinh^2(v)}du\,dv \\
&= \frac{\sqrt{z}}{\pi}\int_0^{\infty}\frac{\sin(\kappa v)\sin\left(\kappa\tanh^{-1}(\sqrt{z}\tanh(v))\right)}{\cosh^2(v)-z\sinh^2(v)}dv\,,
\end{aligned} \tag{3.10}$$

where now, in addition to the residues at $u=\frac{(2n-1)\pi i}{2}\pm\tanh^{-1}(\sqrt{z}\tanh(v))$, there are also residues at $u=\frac{(2n-1)\pi i}{2}$.

In the $z\to 1$ limit we add the integrals in equations (3.9) and (3.10) and neglect terms that behave like $(1-z)\log(1-z)$ to get

$$\rho_{\rm fin}(\kappa) = \lim_{z\to 1}\frac{1}{\pi}\int_0^{\infty}\frac{\cos\left(\kappa\left(v-\tanh^{-1}(\sqrt{z}\tanh(v))\right)\right)}{\cosh^2(v)-z\sinh^2(v)}dv + \frac{1}{2\pi}\log(1-z)\,. \tag{3.11}$$

To evaluate this integral we write $\sqrt{z}=1-\epsilon$, and drop terms which would become irrelevant in the limit

$$\begin{aligned}\rho_{\rm fin}(\kappa) &= \lim_{\epsilon\to 0}\left(\frac{1}{\pi}\int_0^\infty \frac{\cos\left(\frac{\kappa}{2}\log(1+\frac{\epsilon}{2}e^{2v})\right)}{1+\frac{\epsilon}{2}e^{2v}}dv+\frac{1}{2\pi}\log(2\epsilon)\right)\\ &= \lim_{\epsilon\to 0}\frac{1}{2\pi}\left(\int_{\frac{\epsilon}{2}}^\infty \frac{\cos\left(\frac{\kappa\xi}{2}\right)}{e^\xi-1}d\xi+\log(2\epsilon)\right),\end{aligned} \tag{3.12}$$

where $\xi=\log(1+\frac{\epsilon}{2}e^{2v})$. To get rid of the infinities in the limit we calculate

$$\partial_\kappa\rho_{\rm fin}(\kappa)=-\frac{1}{4\pi}\int_0^\infty\frac{\xi\sin\left(\frac{\kappa\xi}{2}\right)}{e^\xi-1}d\xi\,, \tag{3.13}$$

and since $\rho_{\rm fin}(0)=\frac{\log(2)}{\pi}$ (as can be seen from eq. (3.12) or from a direct calculation) we get

$$\rho_{\rm fin}(\kappa)=\frac{\log(2)}{\pi}+\int_0^\kappa(\partial_\kappa\rho_{\rm fin}(\kappa))d\kappa=\frac{4\log(2)-2\gamma-\Psi(\frac{i\kappa}{2})-\Psi(-\frac{i\kappa}{2})}{4\pi}\,, \tag{3.14}$$

where γ is Euler's constant, and Ψ is the digamma (polygamma) function. This result agrees with the numerical estimates of [5].

4. Mathematical preliminary

In the next section we shall see that in order to calculate the Virasoro generators in the κ basis, we need to introduce the differential operator $\mathcal{L}_\kappa$. This operator is a sum of two shift operators in the imaginary direction. We define and explain the use of this operator in section 4.1.

We shall have to operate with $\mathcal{L}_\kappa$ on delta functions, generating delta functions with complex arguments, creatures that we usually do not encounter. Thus, in section 4.2 we elaborate on the definition of these delta functions and demonstrate their use.

4.1 The operator $\mathcal{L}_\kappa$

Our task in this section is to find an operator $\mathcal{L}_\kappa$ which obeys

$$\mathcal{L}_\kappa(\kappa v_n^\kappa)=nv_n^\kappa\,. \tag{4.1}$$

Equivalently this operator should obey the relation

$$\mathcal{L}_\kappa\frac{\kappa v_n^\kappa}{\sqrt{n}}z^n=z\partial_z\frac{v_n^\kappa}{\sqrt{n}}z^n. \tag{4.2}$$

Summing over n, and using the generating function (2.2), we get

$$\mathcal{L}_\kappa(1-e^{-\kappa u})=\frac{1}{2}\sin(2u)e^{-\kappa u}\,, \tag{4.3}$$

where $u = \tan(z)$. We see that

$$\mathcal{L}_\kappa = \frac{1}{2}\sin(2\partial_\kappa) \tag{4.4}$$

satisfies the above equation.

The operator $\mathcal{L}_\kappa$ is a sum of two shift operators

$$\mathcal{L}_\kappa f(\kappa) = \frac{\exp(2i\partial_\kappa) - \exp(-2i\partial_\kappa)}{4i} f(\kappa) = \frac{f(\kappa+2i) - f(\kappa-2i)}{4i}\,, \tag{4.5}$$

and as such is defined only for functions which can be Taylor expanded with convergence radius $r > 2$. Since $\mathcal{L}_\kappa$ contains only odd powers of ∂_κ we get for functions with convergence radius $r > 2$ on the real axis an integration by parts formula

$$\int_{-\infty}^{\infty} f(\kappa)\mathcal{L}_\kappa g(\kappa) d\kappa = -\int_{-\infty}^{\infty} g(\kappa)\mathcal{L}_\kappa f(\kappa) d\kappa\,. \tag{4.6}$$

The case $r = 2$ should be handled with care. We illustrate it with an example that will be used in section 5.

$$\exp(2i\partial_\kappa)\frac{1}{\cosh\left(\frac{(\kappa-\kappa')\pi}{4}\right)} = \frac{1}{\cosh\left(\frac{(\kappa-\kappa'+2i)\pi}{4}\right)} = \frac{-i}{\sinh\left(\frac{(\kappa-\kappa')\pi}{4}\right)}\,, \tag{4.7}$$

where $\kappa' \neq \kappa$. For $\kappa' = \kappa$ the radius of convergence is $r = 2$, and the above expression is undefined. To extract the singular part of it we write

$$\begin{aligned}
\exp(2i\partial_\kappa)\frac{1}{\cosh\left(\frac{(\kappa-\kappa')\pi}{4}\right)} &= \lim_{\epsilon\to 0}\left(\exp((2-\epsilon)i\partial_\kappa)\frac{1}{\cosh\left(\frac{(\kappa-\kappa')\pi}{4}\right)}\right) \\
&= \lim_{\epsilon\to 0}\frac{-i}{\sinh\left(\frac{(\kappa-\kappa')\pi}{4}\right) - i\epsilon} \\
&= -i\mathcal{P}\frac{1}{\sinh\left(\frac{(\kappa-\kappa')\pi}{4}\right)} + \pi\delta\left(\sinh\left(\frac{(\kappa-\kappa')\pi}{4}\right)\right) \\
&= -i\mathcal{P}\frac{1}{\sinh\left(\frac{(\kappa-\kappa')\pi}{4}\right)} + 4\delta(\kappa-\kappa')\,,
\end{aligned} \tag{4.8}$$

where $\mathcal{P}$ represents the principal value of the function. In a similar fashion

$$\exp(-2i\partial_\kappa)\frac{1}{\cosh\left(\frac{(\kappa-\kappa')\pi}{4}\right)} = i\mathcal{P}\frac{1}{\sinh\left(\frac{(\kappa-\kappa')\pi}{4}\right)} + 4\delta(\kappa-\kappa')\,. \tag{4.9}$$

We can take the symmetric and antisymmetric parts of these shift operators and write

$$\begin{aligned}
\cos(2\partial_\kappa)\left(\frac{1}{\cosh\left(\frac{(\kappa-\kappa')\pi}{4}\right)}\right) &= 4\delta(\kappa-\kappa')\,, \\
\sin(2\partial_\kappa)\left(\frac{1}{\cosh\left(\frac{(\kappa-\kappa')\pi}{4}\right)}\right) &= -\mathcal{P}\frac{1}{\sinh\left(\frac{(\kappa-\kappa')\pi}{4}\right)}\,.
\end{aligned} \tag{4.10}$$

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The antisymmetric part is of importance because of (4.4). The importance of the symmetric part lies in the "trigonometric" identity

$$\sin(2\partial_\kappa)\,(f(\kappa)g(\kappa)) = (\sin(2\partial_\kappa)f(\kappa))\,(\cos(2\partial_\kappa)g(\kappa)) + (\cos(2\partial_\kappa)f(\kappa))\,(\sin(2\partial_\kappa)g(\kappa))\,. \tag{4.11}$$

We shall expand the definition of $\mathcal{L}_\kappa$ to functions whose radius of convergence is $r<2$ at the end of the next subsection.

4.2 The one dimensional delta function with complex argument

In this subsection we present the properties of the complex delta function, as it will emerge when we shall operate with $\mathcal{L}_\kappa$ on $\delta(\kappa-\kappa')$ in the calculations of the Virasoro generators. Formally, $\mathcal{L}_\kappa\delta(\kappa-\kappa')$ is an infinite sum of delta function derivatives, but does this sum converge to a distribution? We demonstrate that this sum is an object very similar to a regular distribution.

So far, we defined the operation of $\mathcal{L}_\kappa$ only on functions with convergence radius $r>2$. However, for the delta function, there is no notion of convergence radius because it is a distribution. We consider the definition of the delta function as a "limit" of functions,

$$\delta(\kappa) = \lim_{\epsilon\to 0}\delta_\epsilon(\kappa)\,. \tag{4.12}$$

Different δ_ϵ sequences have different radii of convergence. While the sequence used in (4.8) has a zero radius, other sequences, such as the limit of gaussians,

$$\delta_\epsilon(\kappa) = \frac{1}{\sqrt{\pi\epsilon}}e^{-\frac{\kappa^2}{\epsilon}}\,, \tag{4.13}$$

are analytic in the whole complex plane. Henceforth, we define the delta function using a sequence of this type. We can now define

$$\mathcal{L}_\kappa\delta(\kappa-\kappa') = \lim_{\epsilon\to 0}\mathcal{L}_\kappa\delta_\epsilon(\kappa-\kappa') \equiv \frac{1}{4i}\left(\delta(\kappa-\kappa'+2i) - \delta(\kappa-\kappa'+2i)\right). \tag{4.14}$$

The complex arguments of the delta functions may seem strange, but in fact integrations involving these delta functions are similar to the familiar case of a real argument. This can be seen using the behaviors at (real directed) infinity of δ_ϵ and contour arguments.[1] These contour arguments only apply when we convolute the delta function with an analytic function $f(\kappa)$ that has no poles on the way to the new contour of integration. Thus, the complex delta function acts as a distribution, when confined to this class of functions.

Suppose now that $f(\kappa)$ has simple poles κ_n in the range $0\le \mathrm{Im}(\kappa)\le 2$. To evaluate $\int_{-\infty}^{\infty}\delta(\kappa-2i)f(\kappa)d\kappa$, we displace the contour to the line $\mathrm{Im}(\kappa)=2$, and pick up the residues along the way. Poles for which $0<\mathrm{Im}(\kappa_n)<2$, contribute $2\pi i\delta(\kappa_n-2i)\mathrm{res}_{\kappa_n}(f(\kappa))$, while poles for which $\mathrm{Im}(\kappa_n)=2$ contribute $\pi i\delta(\kappa_n-2i)\mathrm{res}_{\kappa_n}(f(\kappa))$. The last case to consider is

[1]Note that $z\neq 0$ does not imply $\delta(z)=0$. In fact getting expressions such as $\delta(2i)$ in a result would probably mean that a non-legitimate manipulation was performed on the way. Also note, that while $\delta_\epsilon(\kappa-\kappa'\pm 2i)$ are complex, the combination $\mathcal{L}_\kappa\delta_\epsilon(\kappa-\kappa')$ is a real function.

the case of integrating the principle part of poles located on the real line. In this case we again get a contribution of $\pi i\delta(\kappa_n-2i)\mathrm{res}_{\kappa_n}(f(\kappa))$ to the integral. All in all

$$\int_{-\infty}^{\infty}\delta(\kappa-2i)\mathcal{P}f(\kappa)d\kappa = f(2i)+2\pi i\sum_{0<\mathrm{Im}(\kappa_n)<2}\mathrm{res}_{\kappa_n}(f(\kappa))\delta(\kappa_n-2i)+ \\ +\pi i\sum_{\mathrm{Im}(\kappa_n)=0,2}\mathrm{res}_{\kappa_n}(f(\kappa))\delta(\kappa_n-2i)\,. \tag{4.15}$$

Due to a change in the orientation of integration

$$\int_{-\infty}^{\infty}\delta(\kappa+2i)\mathcal{P}f(\kappa)d\kappa = f(-2i)-2\pi i\sum_{-2<\mathrm{Im}(\kappa_n)<0}\mathrm{res}_{\kappa_n}(f(\kappa))\delta(\kappa_n+2i)- \\ -\pi i\sum_{\mathrm{Im}(\kappa_n)=-2,0}\mathrm{res}_{\kappa_n}(f(\kappa))\delta(\kappa_n+2i)\,. \tag{4.16}$$

We see that unlike regular distributions, the convolution of these generalized distributions with analytic functions can result in generalized distributions, rather then functions. The incorporation of multiple poles is straightforward and adds terms with derivatives of the delta function, but we shall not need it here.

Finally we introduce a recipe for handling expressions such as $\mathcal{L}_\kappa f(\kappa)$, where $f(\kappa)$ has radius of convergence $r<2$

$$\mathcal{L}_\kappa f(\kappa)=\mathcal{L}_\kappa\int\delta(\kappa-\tilde{\kappa})f(\tilde{\kappa})d\tilde{\kappa}\equiv\int\mathcal{L}_\kappa\delta(\kappa-\tilde{\kappa})f(\tilde{\kappa})d\tilde{\kappa}\,. \tag{4.17}$$

This definition involves changing the order of integration with the limit (4.12). As we naturally think of this limit as being taken after all integrations were performed, we shall refer to the r.h.s of this equation as the definition of $\mathcal{L}_\kappa f(\kappa)$. For the case $r>2$ this definition coincides with (4.5).

5. The Virasoro generators in the κ basis

In this section we obtain the form of the Virasoro generators in the κ basis for a single scalar field. Since the expressions are cumbersome, we start in 5.1 by finding L_0, which is the simplest one. It is also the most useful one, in particular when working in the Siegel gauge. In subsection 5.2 we calculate $L_{\pm 1}$ and confirm the closure of the sl(2) algebra directly. Next, we get the general expression for all the generators and obtain identities that follow from the Virasoro algebra. All of the above is done in the zero momentum sector for clarity. We conclude by giving the general expressions for the generators including the zero mode.

5.1 Calculating L_0

Formally, writing the Virasoro generator L_0 in the κ basis involves a simple change of basis

$$L_0=\sum_{n=1}^{\infty}na_n^\dagger a_n=\sum_{n=1}^{\infty}\int_{-\infty}^{\infty}\frac{d\kappa\, d\kappa'\, a_{\kappa'}^\dagger a_\kappa n v_n^\kappa v_n^{\kappa'}}{\sqrt{\mathcal{N}(\kappa)\mathcal{N}(\kappa')}}=\int_{-\infty}^{\infty}\frac{d\kappa\, d\kappa'\, a_{\kappa'}^\dagger a_\kappa}{\sqrt{\mathcal{N}(\kappa)\mathcal{N}(\kappa')}}g_0^{\kappa,\kappa'}\,, \tag{5.1}$$

where we define[2]

$$g_0^{\kappa,\kappa'} \equiv \sum_{n=1}^{\infty} n v_n^{\kappa} v_n^{\kappa'} \,. \tag{5.2}$$

However, it was shown in [9] that this sum is highly divergent. Thus, $g_0^{\kappa,\kappa'}$ cannot be given by a regular function. We derive an analytic expression for it as a generalized distribution by acting on both sides of the completeness relation (2.4) with the $\mathcal{L}_\kappa$ operator

$$\begin{aligned} g_0^{\kappa,\kappa'} = \mathcal{L}_\kappa\left(\kappa\sum_{n=1}^{\infty} v_n^{\kappa} v_n^{\kappa'}\right) &= \mathcal{L}_\kappa\left(\kappa\mathcal{N}(\kappa)\delta(\kappa-\kappa')\right)\\ &= \frac{\sinh\left(\frac{\kappa'\pi}{2}\right)}{2i}\left(\delta(\kappa-\kappa'+2i)-\delta(\kappa-\kappa'-2i)\right)\\ &= \frac{\cosh\left(\frac{(\kappa'+\kappa)\pi}{4}\right)}{2}\left(\delta(\kappa-\kappa'+2i)+\delta(\kappa-\kappa'-2i)\right), \end{aligned} \tag{5.3}$$

where in the last step we used the properties of the delta function in order to give a manifestly symmetric expression for $g_0^{\kappa,\kappa'}$.

We conclude this subsection with a consistency check. Define the covariant metric by the following convergent sum

$$g^0_{\kappa,\kappa'} \equiv \sum_{n=1}^{\infty} \frac{v_n^{\kappa} v_n^{\kappa'}}{n} \,. \tag{5.4}$$

By the completeness relation (2.4) and the definition of $\mathcal{L}_\kappa$ (4.1), $g^0_{\kappa,\kappa'}$ obeys the identity

$$\mathcal{L}_\kappa(\kappa g^0_{\kappa,\kappa'}) = \mathcal{N}(\kappa)\delta(\kappa-\kappa') \,. \tag{5.5}$$

We evaluate this metric by the methods of [11] and obtain

$$g^0_{\kappa,\kappa'} = \frac{2\sinh\left(\frac{\kappa\pi}{4}\right)\sinh\left(\frac{\kappa'\pi}{4}\right)}{\kappa\kappa'\cosh\left(\frac{(\kappa-\kappa')\pi}{4}\right)} \,. \tag{5.6}$$

This expression can be used to verify eq. (5.5) directly

$$\begin{aligned} \mathcal{L}_\kappa(\kappa g^0_{\kappa,\kappa'}) &= \frac{\sinh\left(\frac{\kappa'\pi}{4}\right)}{\kappa'}\left(\sin(2\partial_\kappa)\sinh\left(\frac{\kappa\pi}{4}\right)\right)\left(\cos(2\partial_\kappa)\frac{1}{\cosh\left(\frac{(\kappa-\kappa')\pi}{4}\right)}\right)\\ &= \frac{4\sinh\left(\frac{\kappa'\pi}{4}\right)\cosh\left(\frac{\kappa\pi}{4}\right)\delta(\kappa-\kappa')}{\kappa'} = \mathcal{N}(\kappa)\delta(\kappa-\kappa') \,, \end{aligned} \tag{5.7}$$

where we used eq. (4.10), (4.11), and the fact that

$$\cos(2\partial_\kappa)\sinh\left(\frac{\kappa\pi}{4}\right) = 0 \,, \qquad \sin(2\partial_\kappa)\sinh\left(\frac{\kappa\pi}{4}\right) = \cosh\left(\frac{\kappa\pi}{4}\right). \tag{5.8}$$

Using the complex delta function integration rules (4.15), (4.16), one can verify that the metrics obey

$$\int_{-\infty}^{\infty}\frac{d\tilde\kappa}{\mathcal{N}(\tilde\kappa)} g^0_{\kappa,\tilde\kappa} g_0^{\tilde\kappa,\kappa'} = \mathcal{N}(\kappa)\delta(\kappa-\kappa') \,, \tag{5.9}$$

as expected.

[2]The matrix $g_0^{\kappa,\kappa'}$ will play the role of a contravariant metric in [12].

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5.2 Calculating $L_{\pm 1}$

We use similar methods to obtain the remaining sl(2) generators. Actually, it is enough to find L_{-1}, since L_1 can be obtained by hermitian conjugation. We write L_{-1} in the κ basis

$$\begin{aligned} L_{-1} &= \sum_{n=1}^{\infty} \sqrt{n(n+1)} a_{n+1}^{\dagger} a_n \\ &= \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} \frac{d\kappa\, d\kappa'\, a_{\kappa'}^{\dagger} a_{\kappa}}{\sqrt{\mathcal{N}(\kappa)\mathcal{N}(\kappa')}} \sqrt{n(n+1)} v_n^{\kappa} v_{n+1}^{\kappa'} = \int_{-\infty}^{\infty} \frac{d\kappa\, d\kappa'\, a_{\kappa'}^{\dagger} a_{\kappa}}{\sqrt{\mathcal{N}(\kappa)\mathcal{N}(\kappa')}} g_1^{\kappa,\kappa'}, \end{aligned} \tag{5.10}$$

where

$$g_1^{\kappa,\kappa'} \equiv \sum_{n=1}^{\infty} \sqrt{n(n+1)} v_n^{\kappa} v_{n+1}^{\kappa'}. \tag{5.11}$$

Since the sum does not converge, we define the well behaved expression[3]

$$g^1_{\kappa,\kappa'} \equiv \sum_{n=1}^{\infty} \frac{1}{\sqrt{n(n+1)}} v_n^{\kappa} v_{n+1}^{\kappa'}. \tag{5.12}$$

We will get $g_1^{\kappa,\kappa'}$ using

$$g_1^{\kappa,\kappa'} = \mathcal{L}_{\kappa'} \mathcal{L}_{\kappa} \left(\kappa\kappa' g^1_{\kappa,\kappa'}\right). \tag{5.13}$$

Substituting the integral expression for v_n^{κ} (2.11) we obtain

$$\begin{aligned} g^1_{\kappa,\kappa'} &= i\frac{\mathcal{N}(\kappa)\mathcal{N}(\kappa')}{4\pi^2} \int_{-\infty}^{\infty} \frac{du\, dv\, e^{i(\kappa u+\kappa' v)}}{\cosh^2(u)\cosh^2(v)} \sum_{n=1}^{\infty} (-1)^{n-1} \tanh^{n-1}(u) \tanh^n(v) \\ &= i\frac{\mathcal{N}(\kappa)\mathcal{N}(\kappa')}{4\pi^2} \int_{-\infty}^{\infty} \frac{du\, dv\, e^{i(\kappa u+\kappa' v)} \tanh(v)}{\cosh(u)\cosh(v)\cosh(u+v)} \\ &= \frac{1}{\kappa} - \frac{(\kappa-\kappa')\sinh\left(\frac{\kappa'\pi}{2}\right)}{2\kappa\kappa' \sinh\left(\frac{(\kappa-\kappa')\pi}{4}\right)\cosh\left(\frac{(\kappa-\kappa')\pi}{4}\right)}. \end{aligned} \tag{5.14}$$

A direct calculation using (4.10), (4.11) gives

$$\mathcal{L}_{\kappa}(\kappa g^1_{\kappa,\kappa'}) = \mathcal{P} \frac{\sinh\left(\frac{\kappa'\pi}{2}\right)}{\kappa' \sinh\left(\frac{(\kappa-\kappa')\pi}{2}\right)}. \tag{5.15}$$

As this expression has zero radius of convergence for $\kappa = \kappa'$, we use (4.17) and get

$$\begin{aligned} g_1^{\kappa,\kappa'} &= \mathcal{L}_{\kappa'} \int_{-\infty}^{\infty} d\tilde{\kappa}\, \delta(\tilde{\kappa}-\kappa') \mathcal{P} \frac{\sinh\left(\frac{\tilde{\kappa}\pi}{2}\right)}{\sinh\left(\frac{(\kappa-\tilde{\kappa})\pi}{2}\right)} \\ &= \int_{-\infty}^{\infty} d\tilde{\kappa}\, \mathcal{P} \frac{(\delta(\tilde{\kappa}-\kappa'-2i) - \delta(\tilde{\kappa}-\kappa'+2i))\sinh\left(\frac{\tilde{\kappa}\pi}{2}\right)}{4i \sinh\left(\frac{(\kappa-\tilde{\kappa})\pi}{2}\right)} \\ &= -\frac{\sinh\left(\frac{\kappa\pi}{2}\right)}{2} (\delta(\kappa-\kappa'+2i) + \delta(\kappa-\kappa'-2i) + 2\delta(\kappa-\kappa')), \end{aligned} \tag{5.16}$$

[3]Note that unlike $g_0^{\kappa,\kappa'}, g^0_{\kappa,\kappa'}$ which play the role of metrics in suitable spaces, $g_1^{\kappa,\kappa'}, g^1_{\kappa,\kappa'}$ have no such interpretation. In fact they are not even symmetric with respect to κ, κ'.

where in the last step we used the integration rules (4.15), (4.16).

To verify that the sl(2) algebra indeed holds, we write

$$
\begin{aligned}
[L_1, L_{-1}] &= \int_{-\infty}^{\infty} \frac{d\kappa_1\, d\kappa_2\, d\kappa_3\, d\kappa_4\ \sinh\left(\frac{\kappa_1\pi}{2}\right)\sinh\left(\frac{\kappa_3\pi}{2}\right)}{4\sqrt{\mathcal{N}(\kappa_1)\mathcal{N}(\kappa_2)\mathcal{N}(\kappa_3)\mathcal{N}(\kappa_4)}}[a^\dagger_{\kappa_1}a_{\kappa_2}, a^\dagger_{\kappa_4}a_{\kappa_3}] \times \\
&\qquad \times \left(\delta(\kappa_1-\kappa_2+2i) + \delta(\kappa_1-\kappa_2-2i) + 2\delta(\kappa_1-\kappa_2)\right) \times \\
&\qquad \times \left(\delta(\kappa_3-\kappa_4+2i) + \delta(\kappa_3-\kappa_4-2i) + 2\delta(\kappa_3-\kappa_4)\right) \\
&= \int_{-\infty}^{\infty} \frac{d\kappa\, d\kappa'\, d\tilde\kappa\, a^\dagger_{\kappa'}a_\kappa}{4\sqrt{\mathcal{N}(\kappa)\mathcal{N}(\kappa')}} \frac{\sinh\left(\frac{\kappa\pi}{2}\right)\sinh\left(\frac{\kappa'\pi}{2}\right) - \sinh^2\left(\frac{\tilde\kappa\pi}{2}\right)}{\mathcal{N}(\tilde\kappa)} \times \\
&\qquad \times \left(\delta(\kappa-\tilde\kappa+2i) + \delta(\kappa-\tilde\kappa-2i) + 2\delta(\kappa-\tilde\kappa)\right) \times \\
&\qquad \times \left(\delta(\kappa'-\tilde\kappa+2i) + \delta(\kappa'-\tilde\kappa-2i) + 2\delta(\kappa'-\tilde\kappa)\right) = 2L_0\,.
\end{aligned} \tag{5.17}
$$

The derivation of the other commutation relations is similar.

5.3 General L_n

Our strategy should be familiar by now, with a minor change, the appearance of creation (annihilation) operators bilinears. We write

$$
\begin{aligned}
L_{-m} &= \frac{1}{2}\sum_{n=1}^{m-1}\sqrt{n(m-n)}a^\dagger_n a^\dagger_{m-n} + \sum_{n=1}^{\infty}\sqrt{n(n+m)}a^\dagger_{n+m}a_n \\
&= \frac{1}{2}\int_{-\infty}^{\infty}\frac{d\kappa\, d\kappa'\ a^\dagger_{\kappa'}a^\dagger_\kappa}{\sqrt{\mathcal{N}(\kappa)\mathcal{N}(\kappa')}}h_m^{\kappa,\kappa'} + \int_{-\infty}^{\infty}\frac{d\kappa d\kappa' a^\dagger_{\kappa'}a_\kappa}{\sqrt{\mathcal{N}(\kappa)\mathcal{N}(\kappa')}}g_m^{\kappa,\kappa'}\,, \\
L_m = L^\dagger_{-m} &= \frac{1}{2}\int_{-\infty}^{\infty}\frac{d\kappa\, d\kappa'\ a_{\kappa'}a_\kappa}{\sqrt{\mathcal{N}(\kappa)\mathcal{N}(\kappa')}}h_m^{\kappa,\kappa'} + \int_{-\infty}^{\infty}\frac{d\kappa d\kappa' a^\dagger_{\kappa'}a_\kappa}{\sqrt{\mathcal{N}(\kappa)\mathcal{N}(\kappa')}}g_m^{\kappa',\kappa}\,,
\end{aligned} \tag{5.18}
$$

where

$$
\begin{aligned}
g_m^{\kappa,\kappa'} &\equiv \sum_{n=1}^{\infty}\sqrt{n(n+m)}v_n^\kappa v_{n+m}^{\kappa'}\,, \\
h_m^{\kappa,\kappa'} &\equiv \sum_{n=1}^{m-1}\sqrt{n(m-n)}v_n^\kappa v_{m-n}^{\kappa'}\,.
\end{aligned} \tag{5.19}
$$

We start from calculating

$$
\begin{aligned}
g^m_{\kappa,\kappa'} &\equiv \sum_{n=1}^{\infty}\frac{1}{\sqrt{n(n+m)}}v_n^\kappa v_{n+m}^{\kappa'} \\
&= i^m\frac{\mathcal{N}(\kappa)\mathcal{N}(\kappa')}{4\pi^2}\int_{-\infty}^{\infty}\frac{du\,dv\ e^{i(\kappa u+\kappa' v)}}{\cosh^2(u)\cosh^2(v)}\sum_{n=1}^{\infty}(-1)^{n-1}\tanh^{n-1}(u)\tanh^{n+m-1}(v) \\
&= i^m\frac{\mathcal{N}(\kappa)\mathcal{N}(\kappa')}{4\pi^2}\int_{-\infty}^{\infty}\frac{du\,dv\ e^{i(\kappa u+\kappa' v)}\tanh^m(v)}{\cosh(u)\cosh(v)\cosh(u+v)} \\
&= i^{m+1}\frac{\mathcal{N}(\kappa')}{2\pi\kappa}\int_{-\infty}^{\infty}dv\frac{\tanh^{m-1}(v)}{\cosh^2(v)}\left(e^{i(\kappa'-\kappa)v} - e^{i\kappa' v}\right)
\end{aligned} \tag{5.20}
$$

We can now write

$$g_m^{\kappa,\kappa'} = \mathcal{L}_{\kappa'}\mathcal{L}_\kappa(\kappa\kappa' g^m_{\kappa,\kappa'}) = \mathcal{L}_{\kappa'}\mathcal{L}_\kappa \frac{i^{m+1}\kappa'\mathcal{N}(\kappa')}{2\pi}\int_{-\infty}^{\infty} dv \frac{\tanh^{m-1}(v)}{\cosh^2(v)} e^{i(\kappa'-\kappa)v}$$
$$= \kappa'\mathcal{N}(\kappa')\mathcal{L}_{\kappa'}\mathcal{L}_\kappa \frac{v_m^{\kappa'-\kappa}}{\sqrt{m}\mathcal{N}(\kappa'-\kappa)}\,, \tag{5.21}$$

where in the second equality we noticed that the κ independent term drops under the action of the derivative. In the last equality we recognized the integral expression for v_n^κ (2.11), and used the anticommutation relation

$$\big\{\mathcal{L}_{\kappa'}, \kappa'\mathcal{N}(\kappa')\big\} = 0\,. \tag{5.22}$$

In order to continue we recall that $v_m^{\kappa'-\kappa}$ are polynomials with respect to κ, κ', and thus have infinite radius of convergence. Next, we notice that $\mathcal{L}_{\kappa'}\mathcal{L}_\kappa$ acts on a function of $\kappa_- \equiv \kappa' - \kappa$. We define $\kappa_+ \equiv \kappa + \kappa'$, and get

$$\begin{aligned} g_m^{\kappa,\kappa'} &= \frac{\sinh\left(\frac{\kappa'\pi}{2}\right)}{8}(\cos(4\partial_-) - \cos(4\partial_+))\frac{\kappa_- v_m^{\kappa_-}}{\sqrt{m}\sinh\left(\frac{\kappa_-\pi}{2}\right)} \\ &= \frac{\sinh\left(\frac{\kappa'\pi}{2}\right)}{16}\int_{-\infty}^{\infty} d\tilde\kappa\,(\delta(\kappa_- - \tilde\kappa + 4i) + \delta(\kappa_- - \tilde\kappa - 4i) - 2\delta(\kappa_- - \tilde\kappa))\frac{\tilde\kappa v_m^{\tilde\kappa}}{\sqrt{m}\sinh\left(\frac{\tilde\kappa\pi}{2}\right)} \\ &= \sinh\left(\frac{\kappa'\pi}{2}\right)\left(\frac{q_m(\kappa_-)}{\sinh\left(\frac{\kappa_-\pi}{2}\right)} + \frac{i^m\delta(\kappa_- - 2i) - (-i)^m\delta(\kappa_- + 2i)}{2i} - m\sin\left(\frac{m\pi}{2}\right)\delta(\kappa_-)\right), \end{aligned} \tag{5.23}$$

where we used

$$\frac{v_m^{\pm 2i}}{\sqrt{m}} = (\mp i)^{m-1}\,, \qquad \frac{v_m^{\pm 4i}}{\sqrt{m}} = (\mp i)^{m-1} m\,, \tag{5.24}$$

and defined the polynomial

$$q_m(\kappa) \equiv \frac{(\kappa + 4i)v_m^{\kappa+4i} + (\kappa - 4i)v_m^{\kappa-4i} - 2\kappa v_m^\kappa}{16\sqrt{m}} = \oint \frac{e^{-\kappa\tan^{-1}(z)}dz}{z^{m-1}(1+z^2)^2}\,. \tag{5.25}$$

Finally, we find $h_m^{\kappa,\kappa'}$

$$\begin{aligned} h_m^{\kappa,\kappa'} &= \mathcal{L}_{\kappa'}\mathcal{L}_\kappa\left(\kappa\kappa'\sum_{n=1}^{m-1}\frac{1}{(2\pi i)^2\kappa\kappa'}\oint\frac{dz\,dw}{z^{n+1}w^{m-n+1}}e^{-\kappa\tan^{-1}(z) - \kappa'\tan^{-1}(w)}\right) \\ &= \mathcal{L}_{\kappa'}\mathcal{L}_\kappa\oint\frac{dz\,dw}{(2\pi i)^2}\left(\frac{1}{w^{m+1}}\left(\frac{1}{z-w} - \frac{1}{z}\right) + \frac{1}{z^{m+1}}\left(\frac{1}{w-z} - \frac{1}{w}\right)\right)e^{-\kappa\tan^{-1}(z) - \kappa'\tan^{-1}(w)} \\ &= -\mathcal{L}_{\kappa'}\mathcal{L}_\kappa\frac{\kappa_+ v_m^{\kappa_+}}{\sqrt{m}} = q_m(\kappa_+)\,. \end{aligned} \tag{5.26}$$

The Virasoro generators of a single scalar field obey the $c = 1$ Virasoro algebra

$$[L_n, L_m] = (n - m)L_{n+m} + \frac{n^3 - n}{12}\delta_{n+m}\,. \tag{5.27}$$

Substituting the form of the Virasoro generators in the continuous basis (5.18) in the algebra (5.27) gives the identities

$$(n-m)g_{n+m} = [g_n, g_m]\,, \tag{5.28a}$$
$$(n-m)h_{n+m} = h_n g_m + (h_n g_m)^{\mathrm{T}} - h_m g_n - (h_m g_n)^{\mathrm{T}}\,, \tag{5.28b}$$
$$\mathrm{Tr}(h_m h_n) = \frac{1}{6}\delta_{n,m}(n^3-n)\,, \tag{5.28c}$$
$$(n+m)g_{n-m} = g_n g_m^{\mathrm{T}} - g_m^{\mathrm{T}} g_n + h_m h_n \qquad n\geq m\,, \tag{5.28d}$$
$$(n+m)h_{n-m} = g_m h_n + (g_m h_n)^{\mathrm{T}} \qquad n\geq m\,, \tag{5.28e}$$
$$0 = g_n h_m + (g_n h_m)^{\mathrm{T}} \qquad n\geq m\,. \tag{5.28f}$$

In these expressions matrix multiplication is assumed, and the summation of a continuous index $\tilde\kappa$, in the multiplication as well as in the trace, should be understood as integration $\int\frac{d\tilde\kappa}{\mathcal{N}(\tilde\kappa)}$.

To demonstrate the formalism we give a direct proof of (5.28a) and (5.28c). In computing the commutators of g_m we use its form in eq. (5.21) and the expression for v_m^κ in (2.11)

$$\begin{aligned}
[g_n, g_m]^{\kappa\kappa'} &= \kappa'\mathcal{N}(\kappa')\mathcal{L}_\kappa\mathcal{L}_{\kappa'}\int\frac{d\tilde\kappa}{\sqrt{nm}}\tilde\kappa\mathcal{L}_{\tilde\kappa}\left(\frac{v_n^{\tilde\kappa-\kappa}}{\mathcal{N}(\kappa-\tilde\kappa)}\right)\mathcal{L}_{\tilde\kappa}\left(\frac{v_m^{\kappa'-\tilde\kappa}}{\mathcal{N}(\tilde\kappa-\kappa')}\right) - (n\leftrightarrow m)\\
&= \kappa'\mathcal{N}(\kappa')\mathcal{L}_\kappa\mathcal{L}_{\kappa'}\mathcal{L}_\kappa\mathcal{L}_{\kappa'}\int\frac{d\tilde\kappa}{\sqrt{nm}}\tilde\kappa\frac{v_n^{\tilde\kappa-\kappa}v_m^{\kappa'-\tilde\kappa}}{\mathcal{N}(\kappa-\tilde\kappa)\mathcal{N}(\tilde\kappa-\kappa')} - (n\leftrightarrow m)\\
&= \kappa'\mathcal{N}(\kappa')\mathcal{L}_\kappa^2\mathcal{L}_{\kappa'}^2\frac{i^{n+m-1}}{2\pi}\int du\partial_u\left(\frac{e^{-i\kappa u}\tanh^{n-1}u}{\cos^2 u}\right)\frac{e^{i\kappa' u}\tanh^{m-1}u}{\cos^2 u} - (n\leftrightarrow m)\\
&= (n-m)\kappa'\mathcal{N}(\kappa')\mathcal{L}_\kappa\mathcal{L}_{\kappa'}\frac{v_{n+m}^{\kappa'-\kappa}}{\sqrt{n+m}\mathcal{N}(\kappa-\kappa')} = (n-m)g_{n+m}^{\kappa\kappa'}\,,
\end{aligned} \tag{5.29}$$

where in the second step we used the symmetry property $\mathcal{L}_{\tilde\kappa}f(\kappa-\tilde\kappa) = -\mathcal{L}_\kappa f(\kappa-\tilde\kappa)$ and in the third step we performed the integration with respect to $\tilde\kappa$. In the last step we acted with one of the $\mathcal{L}_\kappa$ operators and with one $\mathcal{L}_{\kappa'}$.

For the identity (5.28c) we use generating function techniques. The r.h.s of the identity gives

$$\sum_{n,m\geq 1} z^m w^n \frac{1}{6}\delta_{n,m}(n^3-n) = \frac{z^2w^2}{(1-zw)^4}\,, \tag{5.30}$$

while for the l.h.s

$$\begin{aligned}
\sum_{n,m\geq 1} z^m w^n \int\frac{h_n^{\kappa\kappa'}h_m^{\kappa'\kappa}}{\mathcal{N}(\kappa)\mathcal{N}(\kappa')}d\kappa\, d\kappa' &= \frac{z^2w^2}{(1+z^2)^2(1+w^2)^2}\int\frac{d\kappa\, d\kappa'}{\mathcal{N}(\kappa)\mathcal{N}(\kappa')}e^{-(\kappa+\kappa')(\tan^{-1}z+\tan^{-1}w)}\\
&= \frac{z^2w^2}{(1+z^2)^2(1+w^2)^2}\left(\frac{1}{\cos^2(\tanh^{-1}z+\tanh^{-1}w)}\right)^2\\
&= \frac{z^2w^2}{(1-zw)^4}\,,
\end{aligned} \tag{5.31}$$

where we used (5.26) to sum over n, m.

5.4 The zero mode

Adding the zero mode to the Virasoro generators is simple, as all we have to do is to substitute

$$
\begin{aligned}
L_{-n} &\to L_{-n} + \sqrt{n} a_n^\dagger \sqrt{2\alpha'} p_0 = L_{-n} + \sqrt{2\alpha'} \int d\kappa \frac{\sqrt{n} v_n^\kappa a_\kappa^\dagger}{\sqrt{\mathcal{N}(\kappa)}} p_0 \,, \\
L_n &\to L_n + \sqrt{n} a_n \sqrt{2\alpha'} p_0 + L_n + \sqrt{2\alpha'} \int d\kappa \frac{\sqrt{n} v_n^\kappa a_\kappa}{\sqrt{\mathcal{N}(\kappa)}} p_0 \,, \\
L_0 &\to L_0 + \alpha' p_0^2 \,.
\end{aligned} \tag{5.32}
$$

There are new identities that follow from the Virasoro algebra (5.27)

$$
\begin{aligned}
(n-m)(\sqrt{n+m} v_{n+m}) &= g_m^{\mathrm{T}}(\sqrt{n} v_n) - g_n^{\mathrm{T}}(\sqrt{m} v_m) \,, \\
0 &= g_n(\sqrt{m} v_m) + h_m(\sqrt{n} v_n) \qquad n \geq m \,, \qquad (5.33\text{a}) \\
(n+m)(\sqrt{n-m} v_{n-m}) &= g_m(\sqrt{n} v_n) + h_n(\sqrt{m} v_m) \qquad n \geq m \,. \qquad (5.33\text{b})
\end{aligned}
$$

This completes the derivation of the matter sector Virasoro generators.

6. Conclusions

In this paper we found the finite part of the spectral density and the form of the Virasoro generators in the continuous basis. We hope that it would help solve some of the withstanding problems of string field theory.

The most pretentious goal is to find an analytic solution to string field theory's equation of motion $Q_{BRST}\Psi + \Psi * \Psi = 0$. One way to achieve this might be to follow the analytical methods of [13] in the continuous basis.

It would also be interesting to attempt to calculate string amplitudes analytically. An expression for general string amplitudes was given recently in [14]. The advantage of this expression is that it involves determinants of infinite matrices, making the level truncation calculations much simpler. The matrices involved in this calculation are the Neumann coefficients, coming from the vertices, and the Virasoro operators, coming from the propagators. Using the spectroscopy of the Neumann coefficients, together with our results for the Virasoro operators and the spectral density in the continuous basis, could allow for analytic calculations of string amplitudes.

For performing these tasks some more work is in order. Regarding $\rho_{\mathrm{fin}}(\kappa)$ one still has to understand how to incorporate it in the equations, as there is a discrepancy between CFT results and calculations involving $\rho_{\mathrm{fin}}(\kappa)$ [5]. The analytical expression for the spectral density implies that this is not a numerical artifact. As for the Virasoro generators, the form of the ghost sector is needed. The expressions in the ghost sector resemble those of the matter sector [10]. Thus, they can be calculated using the methods of this paper.

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